Developing a Diagnostic Assessment Instrument for Identifying Students' Understanding of Fraction Equivalence

Objective Measurement in the Social Sciences

Monica Wong
University of Sydney

David Evans
University of Sydney

Judy Anderson
University of Sydney

Contact details: Monica Wong
monica.wong@optusnet.com.au
02 9661 9106

Developing a Diagnostic Assessment Instrument for Identifying Students' Understanding of Fraction Equivalence

Fraction equivalence is a critical concept that is essential for the mastery of common fraction understanding. Assessing the development of fraction equivalence is necessary if teachers are to support students' understanding. A review of fraction studies indicates that assessment instruments developed for particular purposes are rarely adopted by other researchers. In addition, they tend to lack curriculum validity, and may not be supported by technical data from extensive testing. This paper details the process of developing a curriculum referenced pencil-and-paper assessment instrument to measure students' conceptual understanding of fraction equivalence. Following administration of the instrument to students in Years 3 to 5, a Rasch analysis was undertaken to examine issues of item difficulty and fit. The diagnostic properties of many items were further validated through the use of semi-structured interviews with students of differing mathematical achievement levels. Findings from the item-testing phase are discussed, as well as the usefulness of the instrument to determine students' level of understanding of equivalence.

One of the aims of mathematics instruction is for students to develop a deep understanding of key mathematical concepts (Board of Studies NSW [BOS NSW], 2002). Fraction equivalence is one of those concepts. Equivalence imparts the meaning of "worth the same" (Skemp, 1986, p.162), and so two fractions are considered equivalent when they have the same value. In an abstract sense, to fully understand *equivalent* fractions, students need to understand the concept that, "a fraction represents a number that has many names" (Larson, 1980, p.427). For the fraction numeral $\frac{1}{2}$, its equivalence class is represented as $\{\frac{1}{2},\frac{2}{4},\frac{3}{6},\frac{4}{8},...\}$ (Arnon, Nesher & Nirenburg, 2001; Skemp, 1986). Implicit in the concept of equivalence is that items in an equivalence class are interchangeable (Skemp, 1986). So, not only are there an infinite number of fractions between any two fractions, there are also an infinite number of fractions in an equivalence class (Vamvakoussi & Vosniadou, 2004).

Fraction equivalence is a difficult concept for students to understand (Bana, Farrell, & McIntosh, 1997; Pearn, Stephens, & Lewis, 2003). It is a critical concept that is essential for the mastery of fractions, especially fraction addition and subtraction. Bana et al. (1997) found that only one quarter of 12-year-olds could recognise that $\frac{1}{2}=\frac{3}{6}$. Over half the 12-year-olds in their study thought there was no fraction between $\frac{2}{5}$ and $\frac{3}{5}$, while only 7% of those tested said there were "lots" and could correctly name two fractions. Similar results were obtained from the *Success in Numeracy Education* program conducted in Victoria where over half the students were unable to order and add fractions (Pearn et al., 2003).

The new primary mathematics curriculum in NSW has significantly increased the expectations for the early development of fraction knowledge. There is a much greater emphasis on fractions in the *Mathematics K-6 Syllabus* (BOS NSW, 2002) in comparison to the earlier version (NSW Department of School Education, 1989). In addition, students are expected to be able to place (a) halves, quarters and eighths, and (b) fifths and tenths on a number line. Finding equivalent fractions, adding and subtracting simple fractions, and comparing fractions are other inclusions in the syllabus.

Considering the substantial changes in fraction content and the higher learning expectations, further investigation into ways of identifying students' conceptual understanding are required. In particular, it is critical that teachers develop a good understanding of the misconceptions students have about common fractions. The development of a diagnostic instrument would assist teachers, enabling them to readily determine students' levels of understanding.

Literature Review

To develop a deep understanding of common fractions, students need experiences with five sub-constructs or interpretations of common fractions as well as a range of fraction representations. When designing items to assess deep understanding of fractions, these aspects need to be considered. This section of the paper examines relevant literature and implications for this study.

Sub-constructs and Fraction Representations

There are five interconnected sub-constructs or interpretations of fractions as shown in Table 1. These interpretations are both mathematically dependent and psychologically dependent (Kieren, 1980). Conceptual understanding incorporates the ability to make connections within and between these different interpretations, understanding the "sameness" and "distinctness" of the various interpretations (Cathcart, Pothier, Vance & Bezuk, 2006; Kieren, 1980; Lesh, Landau, & Hamilton, 1983). The primary school mathematics curriculum (BOS NSW, 2002) endorses the teaching of fractions using the part/whole, measure, operator and quotient interpretation, whereas ratio is included in the early high school curriculum.

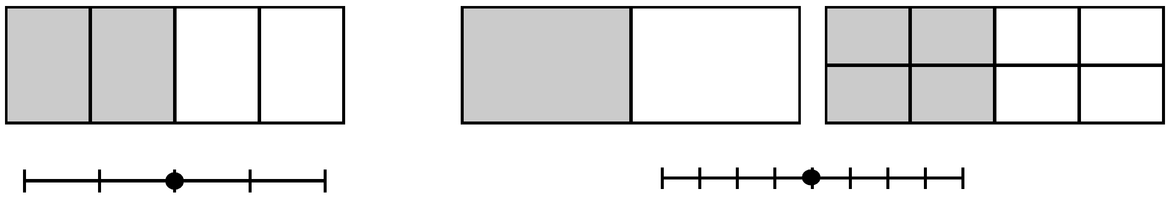
Table 1

Different Fraction Interpretations for the fraction, $\frac{3}{4}$ (Lamon, 2001)

Interpretations	Example
Part/whole	3 out of 4 equal parts of a whole or set of objects
Measure	$\frac{3}{4}$ means a distance of 3 ($\frac{1}{4}$units) from 0 on the number line
Operator	$\frac{3}{4}$ of something, stretching or shrinking
Quotient	3 divided by 4, $\frac{3}{4}$ is the amount each person receives
Ratio	3 parts cement to 4 parts sand

Learning about fractions requires experiences with each of these interpretations as well as with a range of external representations including, a combination of written and spoken symbols, manipulatives, pictures and real world situations (Ball, 1993; Goldin & Shteingold, 2001; Lesh et al., 1983). Written symbols incorporate the formal symbols used for fractions (e.g., $\frac{a}{b}$, where a and b are integers, and b $\neq$ 0) (BOS NSW, 2002). Other written symbols that have special meaning in the domain of fractions include words such as thirds, quarters, halves and so on. The fraction symbol $\frac{a}{b}$, also has spoken counterparts such as "a out of b" or "a over b".

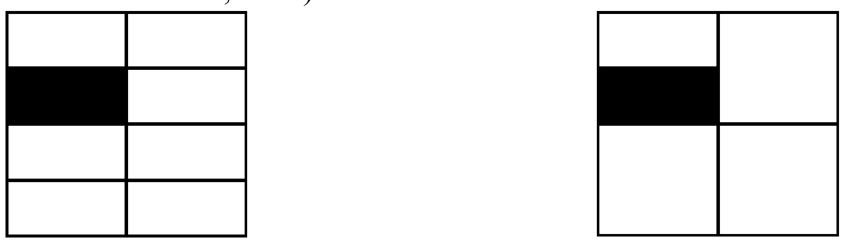
Pictorial fraction representations are classified as simple, equivalent or distractor. Simple representations depict part/whole or measure interpretations in which the number of partitions matches the denominator in a symbolic or verbal representation (Niemi, 1996). Alternatively, equivalent representations occur when the number of partitions of the part/whole or measure representation is a multiplicative factor, less than or greater than, the number of partitions used in the symbolic or verbal description (Niemi, 1996). Figure 1 shows simple and equivalent pictures for the fraction $\frac{2}{4}$ using part/whole and measure representations.



(a) Simple representation (b) Equivalent representations

Figure 1. Simple and equivalent pictorial representations for the fraction two quarters, $\frac{2}{4}$.

Students with unstable fraction understanding are often confused by perceptual distractors (Behr, Lesh, Post & Silver, 1983; Lesh et al., 1983). Figure 2a, depicts an equal area representation where each partition is equivalent, whilst 2b shows an unequal area representation. These different types of representations are used to assist students acquire conceptual understanding of fractions but can also uncover underlying misconceptions (Saxe, Taylor, McIntosh & Gearhart, 2005).



(a) Equal area (b) Unequal area

Figure 2. Equal and unequal area representations

To further support fraction understanding, students are presented with a variety of real life examples such as sharing cookies, pizzas, cakes and chocolate bars. Scenarios such as picnics and restaurant orders are also commonplace. Fruit cut in half and then in quarters is often used to introduce the concept of fractions. Hands on materials are also recommended to represent fractions, and may include paper for folding, fraction circles, paper cutting and string (Ball, 1993; Lamon, 2001; Smith, 2002). Understanding students' usual progress in developing conceptual understanding of fraction equivalence

Assessing the development of fraction equivalence is necessary if teachers are to support students' learning. This encompasses teachers recognising and identifying the stages that students navigate to achieve conceptual understanding of fraction equivalence.

Development of Conceptual Understanding of Fraction Equivalence

Many studies (e.g., Callingham & Watson, 2004; Gould, 2005; Mack, 2005) have examined children's strategy use in the acquisition of fraction knowledge. Some studies have specifically examined addition, comparison and ordering of fractions (e.g., Behr, Wachsmuth, Post & Lesh, 1984; Smith, 1995) and the density of fractions (Vamvakoussi & Vosniadou, 2004). Combining the relevant parts of the literature produces a preliminary developmental path for part/whole, collection and measure interpretations and symbolic representations as shown in Table 2. The stage at which the knowledge and skills appear in the NSW syllabus has also been included. This indicates at what stage, "students typically know and and can do as a consequence of having the syllabus content prescribed" (NSW BOS, 2002, p.14). Typically, Early Stage 1 (ES1) is associated with kindergarten, Stage 1 (S1) with Years 1 and 2, Stage 2 (S2) with Years

3 and 4, and Stage 3 (S3) with Years 5 and 6. Researchers suggest that understanding and representing simple and equivalent fractions does not develop in parallel across the different fraction interpretations, part/whole, collection and measure.

Students make sense of what they are taught by attempting to connect new knowledge and understandings with existing knowledge and experiences (National Research Council [NRC], 2000). When students understand a concept, they formulate "an internal, cognitive representation or mental model that refects the structure of that concept. The representation defines the workspace for problem solving and decision making with respect to the concept" (Halford, 1993, p. 7). Systems of internal representations include "students' personal symbolization constructs, natural language, their visual imagery and spatial representation, their problem-solving strategies and heuristics" (Goldin & Shteingold, 2001, p. 2).

Internal representations cannot be observed directly but access to student's thinking can be achieved through their external representations. When students are asked to express their internal representations they typically use the same mechanisms (e.g., spoken language, pictures, devising real world situations, written symbols and the use of manipulatives) that teachers use to represent the concepts (Goldin & Shteingold, 2001; Sharp, Garofalo, & Adams, 2002; Skemp, 1986). The adequacy of these external representations infers the quality of their conceptual understanding (English & Halford, 1995). Hence, examination of students' external representations can substantiate students' understanding or highlight misconceptions.

Table 2
Development Path for Fraction Understanding.

Level	Part/whole	Collection	Measure	Symbolic
1	Identifies half of a whole object and divides a regular shape into half. (ES1)	Identifies half of a collection of objects. (S1)	Understands that a number line representation shows that numbers measure distances of from zero in terms of some unit distance.(S2)	Able to construct appropriate meanings for fractions in the form of $\frac{a}{b}$.
2	Identifies fractions other than half based on unequal parts. Generates fractions where parts are not equal or there are parts left over.(S1)	Recognises and generates simple representations of fractions of a collection for half and quarter. (S1)	Identifies half of the entire number line. (S2)	Use of a rule or method without understanding, e.g., biggest denominator, bigger the fraction
3	Recognises and generates simple representations of fractions that are generated by repeated halving, e.g., quarters, eighths.(S2)	Recognises and generates simple representations of fractions of a collection for thirds and fifths. (S2)	Identifies location of numbers such as 1 and a half, 2 and a half etc. (S2)	Use of a rule or method with understanding, e.g., ordering fractions and appropriate diagram provided.
4	Recognises and generates other simple representations of fractions, e.g, thirds and fifths. (S3)	Recognises and represents equivalent fractions of a collection. (S2)	Associates the fraction with a point on the number line, where each unit segment has been separated into *b* equal sized segments and the *a*th point to the right of 0.(S2)	
5	Identifies relative size of one geometric region in relation to another.		Recognises and represents equivalent fractions.	
6	Recognises and represents equivalent fractions.(S2)			
7	Co-ordinates part/whole, collection, measure interpretations for simple, equivalent and distractor representations.(S3)			

Assessment of Fraction Understanding

Reliable assessment of students' knowledge is crucial for teachers. It allows teachers to establish and extend students' current level of skill and understanding, as well as address misconceptions by focusing classroom lessons and tasks. The quality of the assessment tool therefore is central to providing reliable data on which to make decisions (Boaler, 1998).

Effective and informative assessment practice requires a variety of assessment strategies that give students multiple opportunities, in varying contexts, to demonstrate what they know, understand and can do in relation to the syllabus outcomes (Department of Education and Training [DET], 2006). Assessment should present mathematics as an interconnected body of knowledge, by engaging students in mathematics that is connected to realistic, illustrative, and pure contexts, thus incorporating the many of representations teachers use (Shannon, 1999). Tasks should accurately and appropriately assess clearly defined aspects of student achievement as well as be time efficient and manageable.

No single assessment tool is available that specifically evaluates students' conceptual understanding of fraction equivalence. In Australia, a number of fraction tests have been used in recent research. Callingham and Watson (2004) devised a 122 item bank that was used to measure Years 3 to 10 students' mental computation competence of fractions, decimals and percentages. A Victorian project, *Success in Numeracy Education* also focussed in part on fraction understanding and developed screening tests for Years 5 to 8 (Pearn et al., 2003). Bana, et al. (1997) and Gould (2005) have devised a number of fraction and decimal items that highlight student error patterns and prevalent misconceptions for students of varying ages. However, these instruments do not seem to have been adopted in other studies and do not report technical data.

This paper outlines the development of a curriculum referenced pencil-and-paper assessment instrument to measure students' conceptual understanding of fraction equivalence. As the assessment is designed for teacher use with the whole class, only pictorial representations, written symbols and real world situations are addressed in the assessment questions.

Methodology

Instrument Design

The assessment items focused on students' understanding of the fraction equivalence using part/whole, collection and measure interpretations. Items also included the use of proper fractions, improper fractions and mixed numerals. Items for the item bank were adapted from previous studies (e.g., Bana et al.,1997; Callingham & Watson, 2004; Pearn et al., 2003) or from released items from international studies such as Third International Mathematics and Science Study (TIMSS), National Assessment of Educational Progress (NAEP). New items were also created so that the overall instrument matched the curriculum expectations for students in Years 3 to 5 in NSW. The overview of the development of the instrument and initial pilot-testing phase is presented in Figure 3.

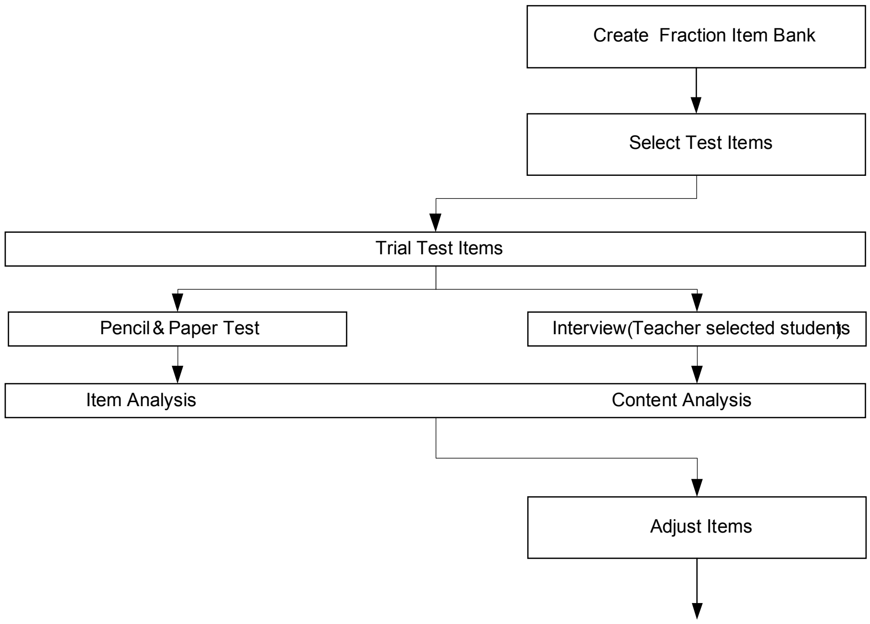


Figure 3. Instrument development pilot.

Over 100 items were assembled for the item bank. Suitable items were selected based on: (a) curriculum relevance; (b) diagnostic potential; (c) item difficulty, so that an individual and a groups of students may be expected to make a variety of responses; (d) suitability for grade level as determined by a group of education professionals (Misailidou & Williams, 2003), and (e) uniqueness of the question. The length of the assessment was also kept to around 40-45 minutes duration to minimise fatigue. The item bank was calibrated through a single test format, consisting of 34 questions, some with multiple parts.

Multiple-choice items were avoided at this stage of item testing in an effort to elicit the full range of possible responses to each item. Students' responses could then be used to identify conceptual errors. Interviews with a selection of students would enable further identification of common errors and problems with item interpretation. The answers during this phase would provide a valuable guide for the development of multiple-choice questions with meaningful distractors.

Test Administration

One hundred and seventy nine students from Years 3 to 5 attending two Sydney metropolitan Catholic systemic primary schools participated in the pilot study. Written permission was obtained from parents or carers prior to the commencement of the study. The participants stratified by their grade level at school appear in Table 3. The *Assessment of Fraction Understanding* test, was administered by the first author and two trained research assistants using standardised procedures with the classroom teacher in attendance. Participants were allowed 45 minutes to complete the assessment and calculators were not permitted.

Table 3
Test Level Scoring Information

Grade level	Sample size (n)	Age (years) Range	Ave.	Gender % Boys	% Girls
3	51	8.15 – 10.2	8.85	47	53
4	58	9.10 – 11.1	9.82	41	49
5	70	10.0 – 11.6	10.78	35	65

Test Analysis

For each item a sample of 50 participant responses were scrutinised. Categories based on the frequency of the response were identified by the first author. These categories became the basis of a coding scheme used for recording the data. Each question was marked correct or incorrect. Non-responses were coded as incorrect as an indication that the student could not answer the question. For questions with multiple parts, each part was coded as an individual item.

Traditional test theory is one method of analysing test data. It measures conceptual understanding of fraction equivalence by summing the number of correct responses in a test. But, a similar change in raw score across different students does not translate to equivalent changes in conceptual understanding of fraction equivalence as questions vary in difficulty. Furthermore, the measure is limited as it is test specific. Rasch modelling overcomes this limitation of traditional test theory by producing estimates of students' level of conceptual understanding that are test independent (Wright & Stone, 1979). The model produces measures students' conceptual understanding of fraction equivalence that take into account the relative difficulty of the questions in the test resulting in measures that are independent of the test items.

While non-responses were coded incorrect, a list of questions with a non-response rate of 10% or greater are listed in Table 4. To examine the difficulty of test items, the test data were subjected to a Rasch analysis using a dichotomous model and the program Quest (Adams & Khoo, 1996).

Interviews

Semi-structured interviews were conducted with nine participants three from each grade level and possessing varying levels of conceptual understanding of fraction equivalence. Participants were asked to solve a number of assessment questions, which were chosen because of question difficulty or because a large number or participants omitted the question. Students were asked to read the question aloud and to describe their method of solution. Probing questions were used to confirm interpretation and to identify common misconceptions. Students were able to use manipulatives such as counters, paper, fraction circles. The interviews were video taped and analysed for common misunderstandings.

Table 4

Questions with 10% or Greater Levels of Non-response

Question	Description	Non respondents n	%
14b	Can you think of another name for the fraction shaded?	41	22.91
18a	What fraction is best represented by point P on the numberline.	50	27.93
18b	What other fraction does it represent.	58	32.40
20a	Use a ruler to mark middle row in tenths.	58	32.40
20b	What fraction did you draw.	38	21.23
20c	Write an equivalent fraction for 3/5	73	40.78
22a	This rectangle represents one whole. What do the following rectangles represent.	46	25.70
22b	Can you think of another name for the fraction shaded?	49	27.37
23	What fraction is shaded?	22	12.29
26b	Show 1 1/2 by marking the number line	41	22.91
26c	Express 1 1/2 as an improper fraction	81	45.25
28a	4/5 = ?/10	33	18.44
28b	6/8 = ?/?	39	21.79
28c	9/12 = ?/4	39	21.79
30a	How are 2/4, 4/8 and 6/12 alike?	36	20.11
30b	Write down three new fractions which represent the same amount	52	29.05
34	The soccer team ordered 7 pizzas	50	27.93

Results

A Rasch analysis was conducted to determine the reliability of the instrument, difficulty and level of fit for each item or assessment question. Overall test reliability was very high with an internal consistency of 0.93. Item difficulties ranged from -4.01 (question 5) to 2.59 (questions 15 and 18b). Reliability of item difficulty estimates is very high, 0.98, suggesting the order of item difficulties would be replicated if the fraction assessment was administered to another group of participants (Bond & Fox, 2001). An examination of the fit statistics shows a number of questions (Table 5) having less compatibility with the Rasch model as standardised infit and outfit statistics were observed to either exceed ±2 (Bond & Fox, 2001). Point biseral coefficients (PBC) have also been included. They provide an indication of item discrimination and facility. It is the correlation between a respondents' responses to a target item (scored dichotomously) and their corresponding total marginal scores (excluding the scored responses to be correlated) (Stenner, 1995). As item facility (percentage correct and incorrect) approaches extreme values of 0% and 100%, PBC decreases. Table 6 contains the item statistics for the remaining items.

Table 5

Item Statistics for the Assessment of Fraction Understanding where there was less compatibility with the Rasch model

Question	Description	Item Difficulty	Error Estimate	Infit MNSQ	Outfit MNSQ	Infit t	Outfit t	Pt Bis
1	Circle the shapes that have been divided in half.	-2.34	0.23	1.31	3.57	2.1	3.6	0.14
7	Draw more balloons so both children in each pair have the same number.	-2.62	0.25	1.35	7.25	2.1	5.5	0.03
8	Here are 15 marbles. Divide your collection into thirds.	-0.18	0.18	1.32	1.66	3.7	3.2	0.35
11	Circle the number lines below where the arrow is pointing to 1/2.	2.53	0.26	1.56	2.58	3	2.6	0.13
14b	Can you think of another name for the fraction shaded?	0.42	0.18	0.75	0.62	-3.2	-2.5	0.70
17d	Write in numerals one and three quarters	0.69	0.19	0.77	0.67	-2.8	-1.9	0.68
24	Luis cut up some apples	-1.29	0.19	1.24	1.86	2.5	2.7	0.31
26a	Draw a shape to represent 1 1/2.	-0.09	0.18	0.85	0.73	-2	-1.7	0.64
29	Circle the fractions that are equal to 1	-0.43	0.18	0.81	0.83	-2.5	-0.9	0.65

Table 6

Item Statistics for the Assessment of Fraction Understanding where the fit was compatible with the Rasch model

Question	Description	Item Difficulty	Error Estimate	Infit MNSQ	Outfit MNSQ	Infit t	Outfit t	Pt Bis
2	Colour in exactly one quarter of these boxed.	-1.87	0.21	1.09	0.93	0.8	-0.1	0.41
3	What fraction of these counters is inside the circle?	-2.00	0.22	0.98	1.11	-0.2	0.4	0.41
4	What fraction has been shaded?	-2.39	0.23	0.94	0.89	-0.4	-0.1	0.40
5	This is a sandwich. Show 3 different ways a sandwich can be cut in half.	-4.01	0.40	1.07	0.97	0.3	0.3	0.24
6	Shade 2/8 of the rectangle.	0.52	0.18	1.02	0.97	0.2	-0.1	0.54
9	Pam cut a pizza into 4 equal pieces. She ate one piece. What fraction of the whole pizza was left?	-0.98	0.19	0.98	0.86	-0.2	-0.6	0.52
10	This is one half the lollies I started with. How many lollies did I start with?	-1.58	0.20	1.11	1.44	1.1	1.4	0.38
12	What fraction has been shaded?	1.51	0.21	1.03	1.20	0.3	0.8	0.48
13	In the figure, how many more squares need to be shaded so that 3/4 of the small squares are shaded?	0.52	0.18	0.88	0.92	-1.4	-0.4	0.61
14a	Shade in 2/2 of the shape below?	0.13	0.18	0.96	1.04	-0.5	0.3	0.56
15	Put a (x) where you think the number 1 would be on the number line.	2.59	0.26	0.96	1.00	-0.2	0.2	0.43
16a	What fraction of the whole bar is A?	0.20	0.18	0.85	0.78	-1.9	-1.4	0.64
16b	What fraction of the whole bar is B?	0.86	0.19	0.85	0.70	-1.7	-1.7	0.64
17a	Write in numerals 6 out of 10	-2.89	0.27	0.90	1.13	-0.5	0.4	0.37
17b	Write in numerals 2 tenths	-1.18	0.19	1.07	1.24	0.8	1	0.43
17c	Write in numerals 5 eighths	-1.18	0.19	1.00	1.20	0.1	0.8	0.46
18a	What fraction is best represented by point P on the numberline	2.01	0.23	0.94	0.98	-0.4	0.1	0.48
18b	What other fraction does it represent.	2.59	0.26	0.87	0.70	-0.8	-0.6	0.47
19a	Write the fraction to represent the pink tulips	-2.69	0.25	0.81	0.61	-1.2	-0.7	0.45
19b	Write another fraction to represent the pink tulips	-0.88	0.18	0.86	0.73	-1.8	-1.3	0.60

Question	Description	Item Difficulty	Error Estimate	Infit MNSQ	Outfit MNSQ	Infit t	Outfit t	Pt Bis
20a	Use a ruler to mark the middle row in tenths	0.82	0.19	0.89	0.81	-1.2	-1	0.60
20b	What fraction did you draw.	1.38	0.20	1.01	0.94	0.1	-0.2	0.52
20c	Write an equivalent fraction for 3/5	1.82	0.22	0.81	0.58	-1.6	-1.5	0.60
21	Terry is baking a cake for her mother.	0.86	0.19	1.14	1.15	1.5	0.8	0.47
22a	This rectangle represents one whole. What do the following rectangles represent.	1.04	0.19	0.98	0.86	-0.2	-0.6	0.55
22b	Can you think of another name for the fraction shaded?	2.40	0.25	1.09	1.12	0.6	0.4	0.41
23	What fraction is shaded?	0.65	0.19	0.97	0.83	-0.4	-0.9	0.57
25	A pie was divided into eighths.	0.42	0.18	1.14	1.18	1.6	1	0.47
26a	Draw a shape to represent 1 1/2.	-0.09	0.18	0.85	0.73	-2	-1.7	0.64
26b	Show 1 1/2 by marking the number line	-0.85	0.18	0.93	0.83	-0.8	-0.8	0.55
26c	Express 1 1/2 as an improper fraction	1.91	0.22	0.84	0.60	-1.2	-1.4	0.58
27	Write fractions in order from smallest to largest	0.04	0.18	0.97	0.93	-0.4	-0.4	0.57
28a	4/5 = ?/10	0.82	0.19	1.00	0.91	0	-0.4	0.54
28b	6/8 = ?/?	0.55	0.18	1.00	0.88	0	-0.6	0.55
28c	9/12 = ?/4	0.72	0.19	0.92	0.84	-0.8	-0.8	0.58
30a	How are 2/4, 4/8 and 6/12 alike?	-0.15	0.18	0.87	0.77	-1.7	-1.4	0.62
30b	Write down three new fractions which represent the same amount	-0.21	0.18	0.97	0.87	-0.4	-0.7	0.57
34	The soccer team ordered 7 pizzas	1.82	0.22	1.19	1.12	1.5	0.5	0.41

As calculated by Quest, comparison of the question difficulty with the average student ability when the question was answered correctly and incorrectly appears in Figure 4. For most questions, the difficulty is located within the student ability range for those students that answered the question incorrectly and those that answered it correctly. However the question difficulty for questions 1, 3, 4, 5, 7, 17a, 19a and 19b are at least 1 logit below the average student ability for those students unable to answer the question correctly. Examination of the item-person map in Figure 5 shows that these questions (bold and underlined), excluding 19b were the easiest questions in the assessment.

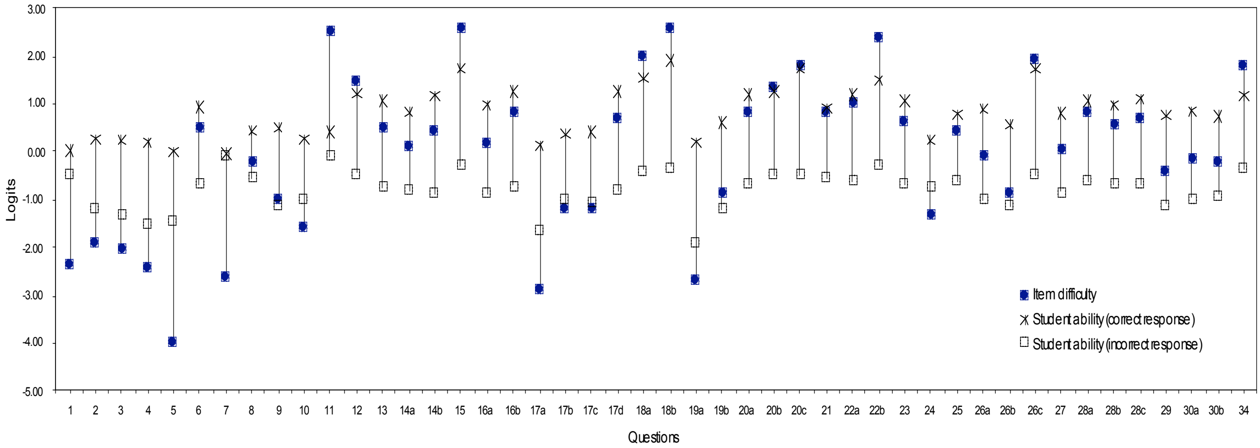


Figure 4. Graph of item difficulty and student abilities for each question.

Further examination of the item-person map in Figure 5 suggests that the assessment contained a range of question difficulties. The person ability map further identifies students by their grade level. In general, Year 5 students appeared at the top of the map (greater conceptual understanding of fraction equivalence), whilst Year 3 students appeared at the bottom (less conceptual understanding of fraction equivalence).

Analysis was conducted to determine if there was any significant difference in item difficulty for the Years 3, 4 and 5 students. Table 7 gives a summary of the person statistics across the grade levels. The mean person ability estimate indicates that Year 3 students found many of the test items difficult, whilst Year 5 students found the more of the test items easier. The reliability of the estimate of person ability is very high for Years 4 and 5.

Table 7

Grade Level Analysis

Grade	*n*	Mean Person Ability	SD	Reliability of Estimate	Item responses All incorrect	Item responses All correct
3	51	-1.13	1.07	.83	26c	
4	58	-0.25	1.44	.92		
5	70	0.77	1.39	.93		5

```
-------------------------------------------------------------------------------------------------
  Person ability estimates       |  Item difficulty estimates
-------------------------------------------------------------------------------------------------
  5.0                            |
                                 |
                             5   |
                                 |
                                 |
                                 |
  4.0                            |
                                 |
                                 |
                            55   |
                                 |
                           455   |
  3.0                            |
                            55   |
                           455   |
                                 |                                    11  15  18b
                           555   |      22b
                            45   |
  2.0                        5   |                                        18a
                           455   |    34  20c                                               26c
                           455   |
                         55555   |      12
                            45   |      20b
                         44455   |
  1.0                     4555   |      22a
                 3444445555555   |                                      16b 21
                       3455555   |      20a                                            17d 28a 28c
                                 |   6  13  23                                               28b
                         44455   |    14b 25
                       4444555   |                                        16a
                       3333555   |      14a                                                  27
  0.0                333445555   |      26a                                                  30a
             33334444455555555   |                      8                                    30b
                        444444   |                                                           29
                            33   |
                      33344555   |
                         33445   |                      19b                26b
 -1.0                   344445   |       9
                         33345   |      24                                                 17b 17c
                      33333345   |
                      33333335   |                      10
                      33334445   |
                           334   |                       2
 -2.0                     3334   |                       3
                          3444   |
                                 |      1  4
                                 |
                          3444   |                      7 19a
                                 |                                                           17a
 -3.0                            |
                                 |
                           333   |
                                 |
                                 |
 -4.0                            |       5
                                 |      P/W          Collect            Measure            Symbol
-------------------------------------------------------------------------------------------------
Each 3,4 or 5 represents 1 student in their respective grades
=================================================================================================
```

Figure 5. Assessment of fraction understanding item difficulty and person ability map.

The items in Figure 5 were further grouped together by the type of fraction interpretation used in the question (i.e. part/whole, collection or measure) or whether the question was presented predominately using written symbols (symbolic). The collection questions were generally easier as students identified the fraction, $\frac{a}{b}$ by probably using the *a* out of *b* strategy. The questions became more difficult when students were required to find an equivalent fraction. Questions that contained the use of symbols ranged in difficulty. The easier questions related to constructing appropriate meaning from fraction words (i.e., 17a) write in numerals 6 out of 10). Questions that students could have applied a rule without understanding were more difficult (i.e., 28 – finding equivalent fractions). Part/whole questions ranged in difficulty, with items using unequal area and equivalent representations most difficult. The easier part/whole questions contained pictorial representations using the fractions halves, quarters and eighths. Measure questions involving number lines were more difficult. Students were able to mark one and a half on the number line easily but were unable to identify fractions on the number line.

Examining each question by grade level showed that question 5 (i.e. This is a sandwich. Show 3 different ways a sandwich can be cut in half) was answered correctly by all Year 5 students, whilst no Year 3 students were able to answer question 26c (i.e. Express $1\frac{1}{2}$ as an improper fraction) correctly. Figure 6 shows the item difficulties for each year level relative to item difficulty for the entire data set. Thus for question 1 (i.e. Circle the shapes that have been divided in half) the question was over 1 logits more difficult for Year 5 students compared to the item difficulty across all grades and was easiest for Year 3 students. Question 11 (i.e., Circle the number lines below where the arrow is pointing to $\frac{1}{2}$) was more difficult for Year 5 students. Item difficulty of questions 9, 10, 13, 14a, 16b and 20a did not vary much across grades.

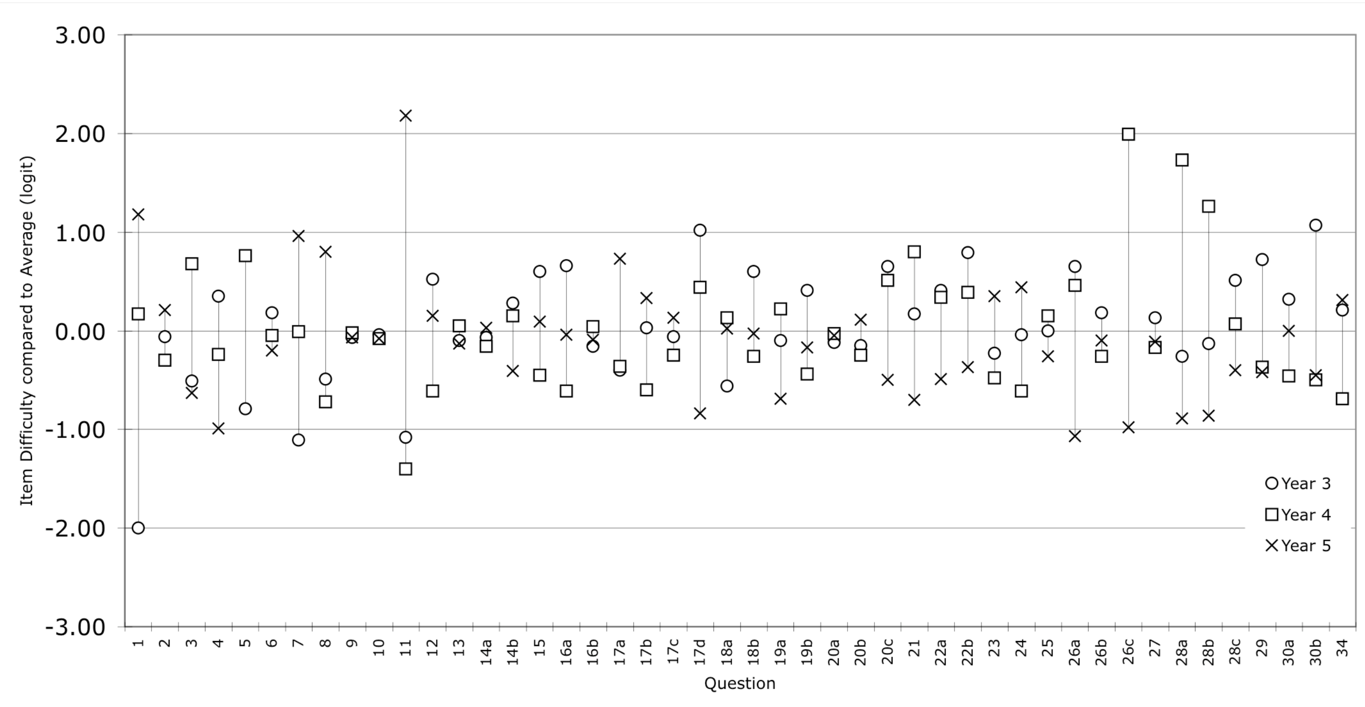


Figure 6. Assessment of fraction understanding item difficulty and person ability map.

Common Errors

Identification of common errors was possible because many questions in the assessment required participants to explain how they calculated their answer. Interviews allowed validation of the questions and to confirm interpretations about what strategies students use to solve them. Generally, students were unable to solve the questions they answered incorrectly in the pencil-and-paper assessment during the interview even with the aid of manipulatives. Scaffolding by the interviewer was the most helpful way of assisting these students. Table 8 contains the common misconceptions.

Table 8
Common errors for particular items

Question	Description	Common Error
6	Shade 2/8 of the rectangle.	Partitioned sections were of unequal size
8	Here are 15 marbles. Divide your collection into thirds.	Groups of three marbles rather than three groups of five marbles.
11	Circle the number lines below where the arrow is pointing to 1/2.	Number line examples where the arrow bisects the section of the number line shown were selected.
12	What fraction has been shaded?	The denominator was calculated from the number of unequal parts shown.
13	In the figure, how many more squares need to be shaded so that 3/4 of the small squares are shaded?	3 squares shaded rather 6 of the 8 squares shown.
14a	Shade in 2/2 of the shape below?	Shading in 2 out of 4 parts of a shape. Fraction has no connection with 1 whole.
16a	What fraction of the whole bar is A?	Fraction given in relation to bar B.
17d	Write in numerals one and three quarters	An incorrect answer of 1/3. Identifying the one and three from the question.
18a	What fraction is best represented by point P on the numberline	Denominator of fraction represents the number of sections of the entire number line drawn, i.e. 6/10.
21	Terry is baking a cake for her mother.	Adding the numerators and denominators of the two fractions together.
22a	This rectangle represents one whole. What do the following rectangles represent.	Adding both rectangles together so the whole is no longer a single rectangle but both.
23	What fraction is shaded?	The denominator was calculated from the number of unequal parts shown.
26a	Draw a shape to represent 1 1/2.	A single partitioned shape was drawn. The wording of the question needs reviewing.
26c	Express 1 1/2 as an improper fraction	The meaning of improper fraction is not known.
27	Write fractions in order from smallest to largest	Comparing denominators, hence the smallest denominator was the smallest fraction.
28a	4/5 = ?/10	Numerator was one less than the denominator, hence 9/10.
28b	6/8 = ?/?	The numerator and denominator differed by 2, e.g., 8/10 or 4/6.
28c	9/12 = ?/4	Numerator was three less than the denominator, hence 1/4.
29	Circle the fractions that are equal to 1	Circling fractions that contained the whole number 1 including fractional parts.
30a	How are 2/4, 4/8 and 6/12 alike?	Examining the numerator and denominator in isolation, hence the numbers are even. Question needs review.
34	The soccer team ordered 7 pizzas	Denominator of the answer given represents the number of whole pizzas or the number of pizza trays still containing pizza.

Interpreting the symbolic representation of fractions created difficulties for students. Understanding the symbols used to represent fractions, and being able to identify the symbol (e.g., thirds are groups of three, not the whole number usage of position three as third in a race) were common errors. Students' knowledge of fractions for part/whole representations was also lacking. This is challenging since students need to: (a) identify the referent whole (English & Halford, 1995; Smith, 2002); (b) understand that "fractions refer to relationship of the equal parts to the whole unit" (Board of Studies NSW, 2002), and (c) understand the language of fractions (English & Halford, 1995).

Discussion and Conclusion

The *Assessment of Fraction Understanding* instrument contained questions with a range of difficulties, varying degrees of item fit and item facility. This would suggest that some questions need review for both clarity of language, mathematical content, proximity to similar questions and difficulty. The easier questions provided poor discrimination between students with different skill levels and will be revised for the next stage in test development. The preliminary results showed that assessment questions were more difficult for Year 3 students with further analysis necessary to ensure the suitability of questions for the various grade levels. Elimination of questions from the test will depend on the focus of the next version. Although easier questions provided poor discrimination between students with different skill levels these questions may be retained if the focus of the assessment is to identify students with very poor understanding of fraction equivalence.

The interviews confirmed the pencil-and-paper assessment answers matched students thought processes. It also showed that the provision of manipulatives in itself provided little assistance in solving the problem for these students. This is consistent with the findings of Ball (1992). Rather, scaffolding was a much more effective method of assisting students overcome their conceptual obstacles.

Overall, the initial assessment provided valuable information for refining the next version of the instrument. However, further analysis is needed for non-response items. Non-response items are omitted items, left out intentionally in the middle of the test or not-reached items occurring at the end of a test. A general rule used is that if the last two or more items are blank (counting back from the last item in the test), then the first item that has a no response is identified as the item attempted by the student. The next and following items are then coded as not-reached. Ludlow and O'Leary (1999) suggest that fair comparisons cannot be made if all types of non-responses are coded as incorrect and students are scored as though they attempted all items. Such a strategy may cause aberrant item statistics for the latter items that students fail to reach (may be a hard test). They recommend the implementation of a two-phase IRT estimation procedure that minimises the statistical effects of omitted items and not-reached items. The first phase incorporates the estimation of item parameters by coding omitted items as incorrect and not-reached items as not administered or missing. During the second phase, item calibrations from the first phase are anchored. Not-reached items are recoded as incorrect and student abilities estimated.

This paper details the initial development and analysis of a pencil and paper instrument to measure conceptual understanding of fraction equivalence. As a result, further consideration needs to be given to: (a) estimating item difficulty and person ability based on Ludlow and O'Leary's (1999) recommended strategy and revising the results; (b) separation of the assessment into grade appropriate levels with appropriate link items, and (c) conducting the assessment over multiple sessions to reduce the effects of fatigue. Following this, revised versions of the instrument will be presented to teachers for feedback and further development, and then be tested on a larger, representative sample.

References

Adams, R. J., & Khoo, S. T. (1996). *Quest: The interactive test analysis system* [Computer software]. Camberwell, Victoria: Australian Council for Educational Research.

Arnon, I., Nesher, P., & Nirenburg, R. (2001). Where do fractions encounter their equivalents? Can this encounter take place in elementary-school? *International Journal of Computers for Mathematical Learning*, *6*, 167-214.

Ball, D. (1992). Magical Hopes: Manipulatives and the Reform of Math Education. American Educator, 16 (Summer), 14-18, 46-47.

Ball, D. (1993). Halves, pieces, and twoths: Constructing and Using Representational Contexts in Teaching Fractions. In T. P. Carpenter & E. Fennema & T. A. Romberg (Eds.), *Rational Numbers: A Integration of Research* (pp. 157 - 196). Hillsdale, New Jersey: Lawrence Erlbaum Associates, Publishers.

Bana, J., Farrell, B., & McIntosh, A. (1997). Student Error Patterns in Fraction and Decimal Concepts. Paper presented at the People in Mathematics Education. In F. Biddulph & K. Carr, *People in mathematics education*, (pp. 50-57), Proceedings of the 20th Annual Conference of MERGA, Melbourne: Deakin University Press.

Behr, M. J., Lesh, R. A., Post, T. R., & Silver, E. A. (1983). Rational number Concepts. In R. A. Lesh & M. Landau (Eds.), Acquisition of Mathematics Concepts and Processes (pp. 91-126). Orlando: Academic Press, Inc.

Behr, M. J., Wachsmuth, I., Post, T. R., & Lesh, R. A. (1984). Order and Equivalence of Rational Numbers: A Clinical Teaching Experiment. *Journal for Research in Mathematics Education, 15*(5), 323-341.

Boaler, J. (1998). Alternative approaches to teaching, learning and assessing mathematics. *Evaluation and Program Planning*, *21*(2), 129-141.

Board of Studies NSW. (2002). *Mathematics K-6 syllabus*. Sydney: Board of Studies NSW.

Bond, T. G., & Fox, C. M. (2001). Applying the Rasch Model: Fundamental Measurement in the Human Sciences. Mahwah: Erlbaum.

Callingham, R., & Watson, J. (2004). A developmental scale of mental computation with part-whole numbers. *Mathematics Education Research Journal, 16*(2), 69-86.

Cathcart, W. G., Pothier, Y., Vance, J. H., & Bezuk, N. S. (2006). *Learning Mathematics in Elementary and Middle Schools: A Learner-Centred Approach* (4th ed., Multimedia ed.). Upper Saddle River: Pearson Prentice Hall.

Department of Education and Training. (2006). Principles for assessment and reporting in NSW government schools. www.det.nsw.gov.au. Accessed: 17 April 2006.

English, L. D., & Halford, G. S. (1995). *Mathematics Education: Models and Processes*. Mahwah: Lawrence Erlbaum Associates.

Goldin, G., & Shteingold, N. (2001). Systems of Representations and the Development of Mathematical Concepts. In A. Cuoco & F. R. Curcio (Eds.), *The Roles of Representation in School Mathematics* (pp. 1-23). Reston: The National Council of Teachers of Mathematics.

Gould, P. (2005). How do you know? The problem of the mathematics dis-ease. *Reflections*, *31*(2), 13-16.

Halford, G. S. (1993). *Children's Understanding: The Development of Mental Models*. Hillsdale: Lwrence Erlbaum Associates.

Jitendra, A. K. (2001). Textbook evaluation and modification for students with learning problems: Part one. *Reading and Writing Quarterly*, *17*(1), 1 - 3.

Kieren, T. (1980). The Rational Number Construct: Its Elements and Mechanisms. In T. E. Kieren (Ed.), *Recent Research on Number Learning* (pp. 125-149). Columbus: ERIC/SMEAC.

Lamon, S. J. (2001). Presenting and Representing: From Fractions to Rational Numbers. In A. Cuoco & F. R. Curcio (Eds.), *The Roles of Representation in School Mathematics* (pp. 146-165). Reston: The National Council of Teachers of Mathematics.

Larson, C. N. (1980). Locating proper fractions on number lines: Effect of length and equivalence. *School Science and Mathematics*, *53*(5), 423-428.

Lesh, R. A., Landau, M., & Hamilton, E. (1983). Conceptual Models and Applied Mathematical Problem-Solving Research. In R. A. Lesh & M. Landau (Eds.), *Acquisition of Mathematics Concepts and Processes* (pp. 263-341). Orlando: Academic Press, Inc.

Ludlow, L. H., & O'Leary, M. (1999). Scoring Omitted and Not-Reached Items: Practical Data Analysis Implications. *Educational and Psychological Measurement*, 59(4), 615-630.

Misailidou, C., & Williams, J. (2003). Diagnostic assessment of children's proportional reasoning. *The Journal of Mathematical Behavior, 22*(3), 335-368.

National Research Council. (2000). *How People Learn: Brain, mind, experience, and school* (Expanded ed.). Washington: National Academy Press.

National Research Council. (2001). *Adding it up: Helping children learn mathematics.* Washington, DC: National Academy Press.

Niemi, D. (1996). *Instuctional Influences on Content Area Explanations and Representational Knowledge: Evidence for the Construct Validity of Measures of Principled Understanding*. CSE Technical Report 403. Los Angeles: National Center for Research on Evaluation, Standards, and Student Testing, University of California.

NSW Department of School Education (1989). *Mathematics K-6.* Sydney: NSW Department of School Education.

Pearn, C., Stephens, M., & Lewis, G. (2003). Assessing rational number knowledge in the middle years of schooling. In C. Australian Association of Mathematics Teachers (Ed.), *Mathematics - Making Waves* (pp. 170-178). Brisbane.

Saxe, G. B., Taylor, E. V., McIntosh, C., & Gearhart, M. (2005). Representing fractions with standard notation: A developmental Analysis. *Journal for Research in Mathematics Education*, *36*(2), 137 - 157.

Shannon, A. (1999). *Keeping Score*. Washington D.C.: National Academies of Science.

Sharp, J., Garofalo, J., & Adams, B. (2002). Children's Development of Meaningful Fraction Algorithms: A Kid's Cookies and a Puppy's Pills. In B. Litwiller & G. Bright (Eds.), *Making Sense of Fractions, Ratios, and Proportions: 2002 Yearbook* (pp. 18-28). Reston: National Council of Teachers of Mathematics.

Skemp, R. R. (1986). *The Pyschology of Learning Mathematics* (2nd ed.). London: Peguin Books.

Smith, J. P. (1995). Competent Reasoning with Rational Numbers. *Cognition and Instruction, 13*(1), 3-50.

Smith, J. P. (2002). The Development of Students' Knowledge of Fractions and Ratios. In B. Litwiller & G. Bright (Eds.), *Making Sense of Fractions, Ratios, and Proportions: 2002 Yearbook* (pp. 3-17). Reston: National Council of Teachers of Mathematics.

Stenner, A.J. (1995) Point-biserial fit indices. *Rasch Measurement Transactions*, 9(1), 416.

Vamvakoussi, X., & Vosniadou, S. (2004). Understanding the structure of the set of rational numbers: a conceptual change approach. *Learning and Instruction*, *14*(5), 453-467.

Wright, B. D., & Stone, M. H. (1979). *Best Test Design*. Chcago: MESA Press.